The initial term $a_0 = e - 1$ cannot be represented exactly in a calculator. Let us call $c$ the approximation of $e - 1$ that we can enter. Verify from the reduction formula (by observing the pattern after a few steps) that

$$a_n = \left[c - \left(\frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right)\right]n!$$

and recall from Equation 10.10.13 that

$$\frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}$$

converges to $e - 1$ as $n \to \infty$. The expression in square brackets converges to $c - (e - 1)$, a nonzero number, which gets multiplied by a fast-growing factor $n!$. We conclude that even if all further calculations (after entering $a_0$) were performed without errors, the initial inaccuracy would cause the computed sequence $\{a_n\}$ to diverge.

**12.** (a) A consolation after the catastrophic outcome of Exercise 11: If we rewrite the reduction formula to read

$$a_{n-1} = \frac{1 + a_n}{n}$$

we can use the inequality used in the squeeze argument to obtain improvements of the approximations of $a_n$. Try $a_{20}$ again using this reverse approach.

(b) We used the reversed reduction formula to calculate quantities for which we have elementary formulas. To see that the idea is even more powerful, develop it for the integrals

$$\int_0^1 x^{n-\theta} e^{1-x}\,dx$$

where $\theta$ is a constant, $0 < \theta < 1$, and $n = 0, 1, \ldots$. For such $\theta$, the integrals are no longer elementary (not solvable in "finite terms"), but the numbers can be calculated quickly. Find the integrals for the particular choice $\theta = \frac{1}{3}$ and $n = 0, 1, \ldots, 5$ to five digits of accuracy.

**13.** An advanced calculator has a key for a peculiar function:

$$E(x) = \begin{cases} 1 & \text{if } x = 0 \\ \dfrac{e^x - 1}{x} & \text{if } x \neq 0 \end{cases}$$

After so many warnings about the subtraction of close numbers, you may appreciate that the definition $\sinh x = \frac{1}{2}(e^x - e^{-x})$ gives inaccurate results for small $x$, where $\sinh x$ is close to $x$. Show that the use of the accurately evaluated function $E(x)$ helps restore the accuracy of $\sinh x$ for small $x$.

## H COMPLEX NUMBERS

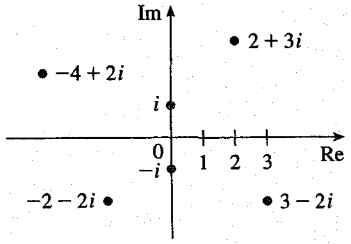


**FIGURE 1**
Complex numbers as points in the Argand plane

A **complex number** can be represented by an expression of the form $a + bi$, where $a$ and $b$ are real numbers and $i$ is a symbol with the property that $i^2 = -1$. The complex number $a + bi$ can also be represented by the ordered pair $(a, b)$ and plotted as a point in a plane (called the Argand plane) as in Figure 1. Thus the complex number $i = 0 + 1 \cdot i$ is identified with the point $(0, 1)$.

The **real part** of the complex number $a + bi$ is the real number $a$ and the **imaginary part** is the real number $b$. Thus the real part of $4 - 3i$ is 4 and the imaginary part is $-3$. Two complex numbers $a + bi$ and $c + di$ are **equal** if $a = c$ and $b = d$, that is, their real parts are equal and their imaginary parts are equal. In the Argand plane the $x$-axis is called the real axis and the $y$-axis is called the imaginary axis.

The sum and difference of two complex numbers are defined by adding or subtracting their real parts and their imaginary parts, respectively:

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

$$(a + bi) - (c + di) = (a - c) + (b - d)i$$

For instance,

$$(1 - i) + (4 + 7i) = (1 + 4) + (-1 + 7)i = 5 + 6i$$

The product of complex numbers is defined so that the usual commutative and distributive laws hold:

$$\begin{aligned}(a + bi)(c + di) &= a(c + di) + (bi)(c + di) \\ &= ac + adi + bci + bdi^2\end{aligned}$$

Since $i^2 = -1$, this becomes

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

**EXAMPLE 1**

$$\begin{aligned}(-1 + 3i)(2 - 5i) &= -(2 - 5i) + 3i(2 - 5i)\\ &= -2 + 5i + 6i - 15(-1) = 13 + 11i\end{aligned}$$ ■

Division of complex numbers is much like rationalizing the denominator of a rational expression. For the complex number $z = a + bi$, we define its **complex conjugate** to be $\bar{z} = a - bi$. To find the quotient of two complex numbers we multiply numerator and denominator by the complex conjugate of the denominator.

**EXAMPLE 2** Express the number $\dfrac{-1 + 3i}{2 + 5i}$ in the form $a + bi$.

SOLUTION We multiply numerator and denominator by the complex conjugate of $2 + 5i$, namely $2 - 5i$, and we take advantage of the result of Example 1:

$$\frac{-1 + 3i}{2 + 5i} = \frac{-1 + 3i}{2 + 5i} \cdot \frac{2 - 5i}{2 - 5i} = \frac{13 + 11i}{2^2 + 5^2} = \frac{13}{29} + \frac{11}{29}i$$ ■

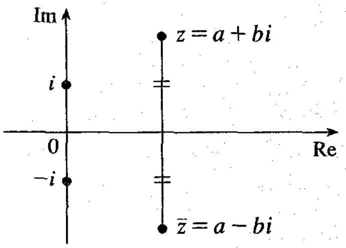


FIGURE 2

The geometric interpretation of the complex conjugate is shown in Figure 2: $\bar{z}$ is the reflection of $z$ in the real axis. We list some of the properties of the complex conjugate in the following box. The proofs follow from the definition and are requested in Exercise 18.

**PROPERTIES OF CONJUGATES**

$$\overline{z + w} = \bar{z} + \bar{w} \qquad \overline{zw} = \bar{z}\,\bar{w} \qquad \overline{z^n} = \bar{z}^n$$

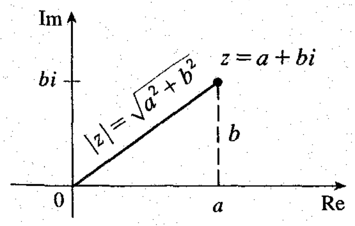


FIGURE 3

The **modulus,** or **absolute value,** $|z|$ of a complex number $z = a + bi$ is its distance from the origin. From Figure 3 we see that if $z = a + bi$, then

$$\boxed{|z| = \sqrt{a^2 + b^2}}$$

Notice that

$$z\bar{z} = (a + bi)(a - bi) = a^2 + abi - abi - b^2i^2 = a^2 + b^2$$

and so

$$\boxed{z\bar{z} = |z|^2}$$

This explains why the division procedure in Example 2 works in general:

$$\frac{z}{w} = \frac{z\bar{w}}{w\bar{w}} = \frac{z\bar{w}}{|w|^2}$$

Since $i^2 = -1$, we can think of $i$ as a square root of $-1$. But we also have $(-i)^2 = i^2 = -1$ and so $-i$ is also a square root of $-1$. We say that $i$ is the **principal square root** of $-1$ and write $\sqrt{-1} = i$. In general, if $c$ is any positive number, we write

$$\sqrt{-c} = \sqrt{c}\, i$$

With this convention the usual derivation and formula for the roots of the quadratic equation

$ax^2 + bx + c = 0$ are valid even when $b^2 - 4ac < 0$:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

EXAMPLE 3 Find the roots of the equation $x^2 + x + 1 = 0$.

SOLUTION Using the quadratic formula, we have

$$x = \frac{-1 \pm \sqrt{1^2 - 4 \cdot 1}}{2} = \frac{-1 \pm \sqrt{-3}}{2} = \frac{-1 \pm \sqrt{3}\, i}{2}$$ ■

We observe that the solutions of the equation in Example 3 are complex conjugates of each other. In general, the solutions of any quadratic equation $ax^2 + bx + c = 0$ with real coefficients $a$, $b$, and $c$ are always complex conjugates. (If $z$ is real, $\bar{z} = z$, so $z$ is its own conjugate.)

We have seen that if we allow complex numbers as solutions, then every quadratic equation has a solution. More generally, it is true that every polynomial equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0$$

of degree at least one has a solution among the complex numbers. This fact is known as the Fundamental Theorem of Algebra and was proved by Gauss.

## POLAR FORM

We know that any complex number $z = a + bi$ can be considered as a point $(a, b)$ and that any such point can be represented by polar coordinates $(r, \theta)$ with $r \geqslant 0$. In fact,

$$a = r\cos\theta \qquad b = r\sin\theta$$

as in Figure 4. Therefore, we have

$$z = a + bi = (r\cos\theta) + (r\sin\theta)i$$

Thus we can write any complex number $z$ in the form

$$\boxed{z = r(\cos\theta + i\sin\theta)}$$

where $\qquad r = |z| = \sqrt{a^2 + b^2} \qquad$ and $\qquad \tan\theta = \dfrac{b}{a}$

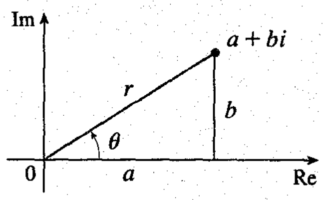


FIGURE 4

The angle $\theta$ is called the **argument** of $z$ and we write $\theta = \arg(z)$. Note that $\arg(z)$ is not unique; any two arguments of $z$ differ by an integer multiple of $2\pi$.

EXAMPLE 4 Write the following numbers in polar form:
(a) $z = 1 + i$ (b) $w = \sqrt{3} - i$

SOLUTION
(a) We have $r = |z| = \sqrt{1 + 1} = \sqrt{2}$ and $\tan\theta = 1$, so we can take $\theta = \pi/4$. Therefore, the polar form is

$$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

(b) Here we have $r = |w| = \sqrt{3 + 1} = 2$ and $\tan\theta = -1/\sqrt{3}$. Since $w$ lies in the fourth quadrant, we take $\theta = -\pi/6$ and

$$w = 2\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$$

The numbers $z$ and $w$ are shown in Figure 5. ■

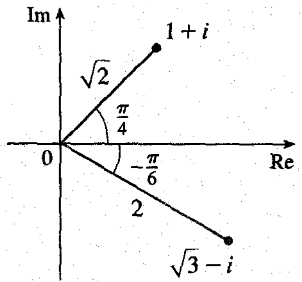


FIGURE 5

The polar form of complex numbers gives insight into multiplication and division. Let

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \qquad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

be two complex numbers written in polar form. Then

$$\begin{aligned} z_1 z_2 &= r_1 r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)] \end{aligned}$$

Therefore, using the addition formulas for cosine and sine, we have

$$\boxed{z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]} \tag{1}$$

This formula says that *to multiply two complex numbers we multiply the moduli and add the arguments.* (See Figure 6.)

A similar argument using the subtraction formulas for sine and cosine shows that *to divide two complex numbers we divide the moduli and subtract the arguments.*

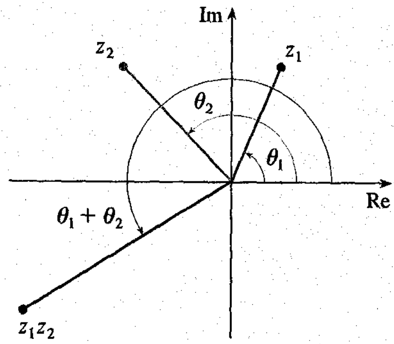


FIGURE 6

$$\boxed{\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)] \qquad z_2 \neq 0}$$

In particular, taking $z_1 = 1$ and $z_2 = z$, we have the following, which is illustrated in Figure 7.

$$\boxed{\text{If} \quad z = r(\cos\theta + i\sin\theta), \quad \text{then} \quad \frac{1}{z} = \frac{1}{r}(\cos\theta - i\sin\theta).}$$

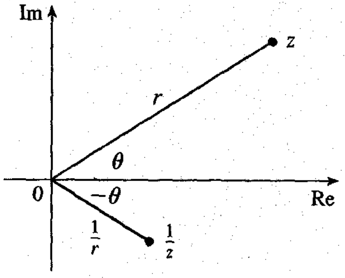


FIGURE 7

**EXAMPLE 5** Find the product of the complex numbers $1 + i$ and $\sqrt{3} - i$ in polar form.

SOLUTION From Example 4 we have

$$1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

and

$$\sqrt{3} - i = 2\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$$

So, by Equation 1,

$$\begin{aligned} (1 + i)(\sqrt{3} - i) &= 2\sqrt{2}\left[\cos\left(\frac{\pi}{4} - \frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{4} - \frac{\pi}{6}\right)\right] \\ &= 2\sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right) \end{aligned}$$

This is illustrated in Figure 8. ■

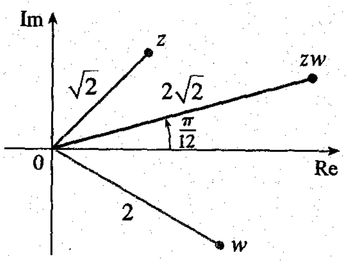


FIGURE 8

Repeated use of Formula 1 shows how to compute powers of a complex number. If

$$z = r(\cos\theta + i\sin\theta)$$

then

$$z^2 = r^2(\cos 2\theta + i\sin 2\theta)$$

and

$$z^3 = zz^2 = r^3(\cos 3\theta + i\sin 3\theta)$$

In general, we obtain the following result, which is named after the French mathematician Abraham De Moivre (1667–1754).

> **(2) DE MOIVRE'S THEOREM** If $z = r(\cos\theta + i\sin\theta)$ and $n$ is a positive integer, then
>
> $$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta)$$

This says that *to take the* n*th power of a complex number we take the* n*th power of the modulus and multiply the argument by* n.

**EXAMPLE 6** Find $(\frac{1}{2} + \frac{1}{2}i)^{10}$.

SOLUTION Since $\frac{1}{2} + \frac{1}{2}i = \frac{1}{2}(1 + i)$, it follows from Example 4(a) that $\frac{1}{2} + \frac{1}{2}i$ has the polar form

$$\tfrac{1}{2} + \tfrac{1}{2}i = \frac{\sqrt{2}}{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$$

So by De Moivre's Theorem,

$$\begin{aligned}(\tfrac{1}{2} + \tfrac{1}{2}i)^{10} &= \left(\frac{\sqrt{2}}{2}\right)^{10}\left(\cos\frac{10\pi}{4} + i\sin\frac{10\pi}{4}\right)\\ &= \frac{2^5}{2^{10}}\left(\cos\frac{5\pi}{2} + i\sin\frac{5\pi}{2}\right) = \frac{1}{32}i\end{aligned}$$

■

De Moivre's Theorem can also be used to find the $n$th roots of complex numbers. An $n$th root of the complex number $z$ is a complex number $w$ such that

$$w^n = z$$

Writing these two numbers in trigonometric form as

$$w = s(\cos\phi + i\sin\phi) \quad \text{and} \quad z = r(\cos\theta + i\sin\theta)$$

and using De Moivre's Theorem, we get

$$s^n(\cos n\phi + i\sin n\phi) = r(\cos\theta + i\sin\theta)$$

The equality of these two complex numbers shows that

$$s^n = r \quad \text{or} \quad s = r^{1/n}$$

and

$$\cos n\phi = \cos\theta \quad \text{and} \quad \sin n\phi = \sin\theta$$

From the fact that sine and cosine have period $2\pi$ it follows that

$$n\phi = \theta + 2k\pi \quad \text{or} \quad \phi = \frac{\theta + 2k\pi}{n}$$

Thus

$$w = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

Since this expression gives a different value of $w$ for $k = 0, 1, 2, \ldots, n-1$, we have the following:

**(3) ROOTS OF A COMPLEX NUMBER** Let $z = r(\cos\theta + i\sin\theta)$ and let $n$ be a positive integer. Then $z$ has the $n$ distinct $n$th roots

$$w_k = r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

where $k = 0, 1, 2, \ldots, n-1$.

Notice that each of the $n$th roots of $z$ has modulus $|w_k| = r^{1/n}$. Thus all the $n$th roots of $z$ lie on the circle of radius $r^{1/n}$ in the complex plane. Also, since the argument of each successive $n$th root exceeds the argument of the previous root by $2\pi/n$, we see that the $n$th roots of $z$ are equally spaced on this circle.

EXAMPLE 7 Find the six sixth roots of $z = -8$ and graph these roots in the complex plane.

SOLUTION In trigonometric form, $z = 8(\cos\pi + i\sin\pi)$. Applying Equation 3 with $n = 6$, we get

$$w_k = 8^{1/6}\left(\cos\frac{\pi + 2k\pi}{6} + i\sin\frac{\pi + 2k\pi}{6}\right)$$

We get the six sixth roots of $-8$ by taking $k = 0, 1, 2, 3, 4, 5$ in this formula:

$$w_0 = 8^{1/6}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \sqrt{2}\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$

$$w_1 = 8^{1/6}\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = \sqrt{2}\, i$$

$$w_2 = 8^{1/6}\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = \sqrt{2}\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$

$$w_3 = 8^{1/6}\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) = \sqrt{2}\left(-\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$$

$$w_4 = 8^{1/6}\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = -\sqrt{2}\, i$$

$$w_5 = 8^{1/6}\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right) = \sqrt{2}\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$$

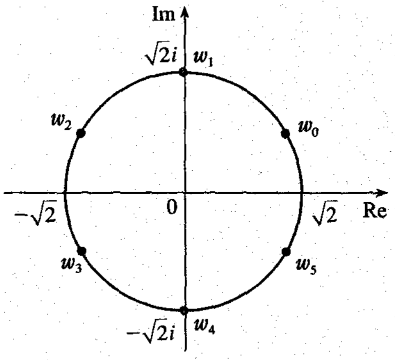


FIGURE 9
The six sixth roots of $z = -8$

All these points lie on the circle of radius $\sqrt{2}$ as shown in Figure 9. ■

## COMPLEX EXPONENTIALS

We also need to give a meaning to the expression $e^z$ when $z = x + iy$ is a complex number. The theory of infinite series as developed in Chapter 10 can be extended to the case where the terms are complex numbers. Using the Taylor series for $e^x$ (10.10.12) as our guide, we define

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \tag{4}$$

and it turns out that this complex exponential function has the same properties as the real exponential function. In particular, it is true that

$$e^{z_1+z_2} = e^{z_1}e^{z_2} \tag{5}$$

If we put $z = iy$, where $y$ is a real number, in Equation 4, and use the facts that

$$i^2 = -1, \quad i^3 = i^2 i = -i, \quad i^4 = 1, \quad i^5 = i, \quad \ldots$$

we get

$$\begin{aligned} e^{iy} &= 1 + iy + \frac{(iy)^2}{2!} + \frac{(iy)^3}{3!} + \frac{(iy)^4}{4!} + \frac{(iy)^5}{5!} + \cdots \\ &= 1 + iy - \frac{y^2}{2!} - i\frac{y^3}{3!} + \frac{y^4}{4!} + i\frac{y^5}{5!} + \cdots \\ &= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \frac{y^6}{6!} + \cdots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right) \\ &= \cos y + i \sin y \end{aligned}$$

Here we have used the Taylor series for $\cos y$ and $\sin y$ (Equations 10.10.16 and 10.10.15). The result is a famous formula called **Euler's formula:**

$$\boxed{e^{iy} = \cos y + i \sin y} \tag{6}$$

Combining Euler's formula with Equation 5, we get

$$e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y) \tag{7}$$

**EXAMPLE 8** Evaluate: (a) $e^{i\pi}$ (b) $e^{-1+i\pi/2}$

SOLUTION

(a) From Euler's equation (6) we have

$$e^{i\pi} = \cos \pi + i \sin \pi = -1 + i(0) = -1$$

(b) Using Equation 7 we get

$$e^{-1+i\pi/2} = e^{-1}\left(\cos \frac{\pi}{2} + i \sin \frac{\pi}{2}\right) = \frac{1}{e}[0 + i(1)] = \frac{i}{e} \qquad \blacksquare$$

Finally, we note that Euler's equation provides us with an easier method of proving De Moivre's Theorem:

$$[r(\cos\theta + i \sin\theta)]^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n(\cos n\theta + i \sin n\theta)$$

## EXERCISES H

**1–14** ■ Evaluate the expression and write your answer in the form $a + bi$.

**1.** $(3 + 2i) + (7 - 3i)$

**2.** $(1 + i) - (2 - 3i)$

**3.** $(3 - i)(4 + i)$

**4.** $(4 - 7i)(1 + 3i)$

**5.** $\overline{12 + 7i}$

**6.** $\overline{2i(\frac{1}{2} - i)}$

**7.** $\dfrac{2 + 3i}{1 - 5i}$

**8.** $\dfrac{5 - i}{3 + 4i}$

**9.** $\dfrac{1}{1 + i}$

**10.** $\dfrac{3}{4 - 3i}$

**11.** $i^3$

**12.** $i^{100}$

**13.** $\sqrt{-25}$ **14.** $\sqrt{-3}\,\sqrt{-12}$

**15–17** ■ Find the complex conjugate and the modulus of each number.

**15.** $3 + 4i$ **16.** $\sqrt{3} - i$ **17.** $-4i$

**18.** Prove the following properties of complex numbers.
(a) $\overline{z + w} = \overline{z} + \overline{w}$
(b) $\overline{zw} = \overline{z}\,\overline{w}$
(c) $\overline{z^n} = \overline{z}^n$, where $n$ is a positive integer
[*Hint:* Write $z = a + bi$, $w = c + di$.]

**19–24** ■ Find all solutions of the equation.

**19.** $4x^2 + 9 = 0$ **20.** $x^4 = 1$

**21.** $x^2 - 8x + 17 = 0$ **22.** $x^2 - 4x + 5 = 0$

**23.** $z^2 + z + 2 = 0$ **24.** $z^2 + \frac{1}{2}z + \frac{1}{4} = 0$

**25–28** ■ Write the number in polar form with argument between 0 and $2\pi$.

**25.** $-3 + 3i$ **26.** $1 - \sqrt{3}\,i$

**27.** $3 + 4i$ **28.** $8i$

**29–32** ■ Find polar forms for $zw$, $z/w$, and $1/z$ by first putting $z$ and $w$ into polar form.

**29.** $z = \sqrt{3} + i, \quad w = 1 + \sqrt{3}\,i$

**30.** $z = 4\sqrt{3} - 4i, \quad w = 8i$

**31.** $z = 2\sqrt{3} - 2i, \quad w = -1 + i$

**32.** $z = 4(\sqrt{3} + i), \quad w = -3 - 3i$

**33–36** ■ Find the indicated power using De Moivre's Theorem.

**33.** $(1 + i)^{20}$ **34.** $(1 - \sqrt{3}\,i)^5$

**35.** $(2\sqrt{3} + 2i)^5$ **36.** $(1 - i)^8$

**37–40** ■ Find the indicated roots. Sketch the roots in the complex plane.

**37.** The eighth roots of 1 **38.** The fifth roots of 32

**39.** The cube roots of $i$ **40.** The cube roots of $1 + i$

**41–46** ■ Write the number in the form $a + bi$.

**41.** $e^{i\pi/2}$ **42.** $e^{2\pi i}$ **43.** $e^{i3\pi/4}$

**44.** $e^{-i\pi}$ **45.** $e^{2+i\pi}$ **46.** $e^{1+2i}$

**47.** Use De Moivre's Theorem with $n = 3$ to express $\cos 3\theta$ and $\sin 3\theta$ in terms of $\cos\theta$ and $\sin\theta$.

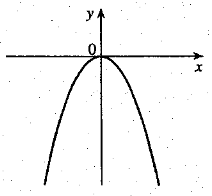
**48.** Use Euler's formula to prove the following formulas for $\cos x$ and $\sin x$:

$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

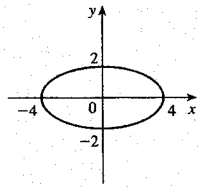
$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

**49.** If $u(x) = f(x) + ig(x)$ is a complex-valued function of a real variable $x$ and the real and imaginary parts $f(x)$ and $g(x)$ are differentiable functions of $x$, then the derivative of $u$ is defined to be $u'(x) = f'(x) + ig'(x)$. Use this together with Equation 7 to prove that if $F(x) = e^{rx}$, then $F'(x) = re^{rx}$ when $r = a + bi$ is a complex number.

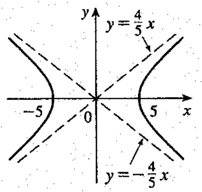
**11.** Parabola



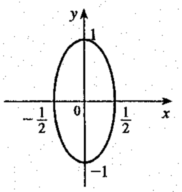
**13.** Ellipse



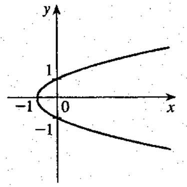
**15.** Hyperbola



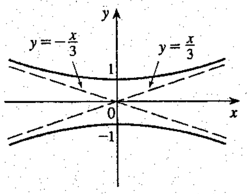
**17.** Ellipse



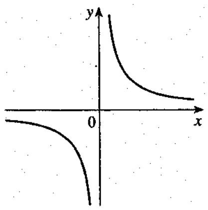
**19.** Parabola



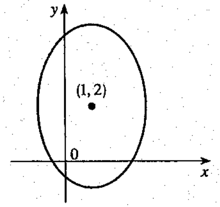
**21.** Hyperbola



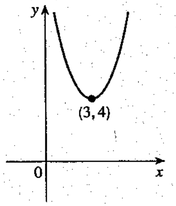
**23.** Hyperbola



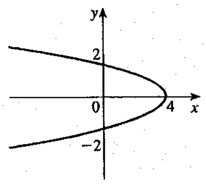
**25.** Ellipse



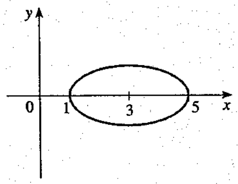
**27.** Parabola



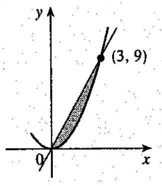
**29.** Parabola



**31.** Ellipse

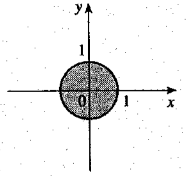


**33.**

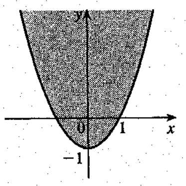


**35.** $y = x^2 - 2x$

**37.**



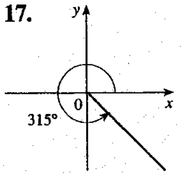
**39.**



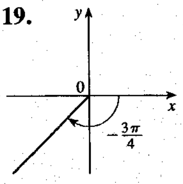
### Exercises D ▪ page A30

**1.** $7\pi/6$ **3.** $\pi/20$ **5.** $5\pi$ **7.** 720° **9.** 75°
**11.** −67.5° **13.** $3\pi$ cm **15.** $\frac{2}{3}$ rad $= (120/\pi)°$

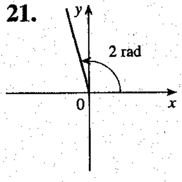
**17.**



**19.**



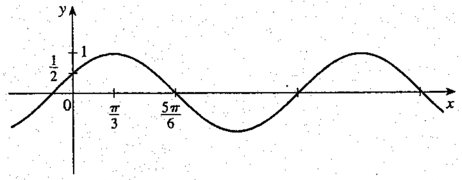
**21.**



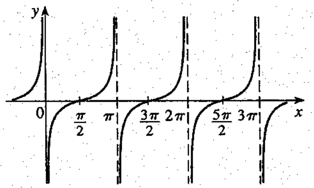
**23.** $\sin(3\pi/4) = 1/\sqrt{2}$, $\cos(3\pi/4) = -1/\sqrt{2}$, $\tan(3\pi/4) = -1$, $\csc(3\pi/4) = \sqrt{2}$, $\sec(3\pi/4) = -\sqrt{2}$, $\cot(3\pi/4) = -1$
**25.** $\sin(9\pi/2) = 1$, $\cos(9\pi/2) = 0$, $\csc(9\pi/2) = 1$, $\cot(9\pi/2) = 0$, $\tan(9\pi/2)$ and $\sec(9\pi/2)$ undefined
**27.** $\frac{1}{2}$, $-\sqrt{3}/2$, $-1/\sqrt{3}$, 2, $-2/\sqrt{3}$, $-\sqrt{3}$

**29.** $\cos\theta = \frac{4}{5}$, $\tan\theta = \frac{3}{4}$, $\csc\theta = \frac{5}{3}$, $\sec\theta = \frac{5}{4}$, $\cot\theta = \frac{4}{3}$
**31.** $\sin\phi = \sqrt{5}/3$, $\cos\phi = -\frac{2}{3}$, $\tan\phi = -\sqrt{5}/2$, $\csc\phi = 3/\sqrt{5}$, $\cot\phi = -2/\sqrt{5}$
**33.** $\sin\beta = -1/\sqrt{10}$, $\cos\beta = -3/\sqrt{10}$, $\tan\beta = \frac{1}{3}$, $\csc\beta = -\sqrt{10}$, $\sec\beta = -\sqrt{10}/3$
**35.** 5.73576 cm **37.** 24.62147 cm **59.** $(4 + 6\sqrt{2})/15$
**61.** $(3 + 8\sqrt{2})/15$ **63.** $\frac{24}{25}$ **65.** $\pi/3$, $5\pi/3$
**67.** $\pi/4$, $3\pi/4$, $5\pi/4$, $7\pi/4$ **69.** $\pi/6$, $\pi/2$, $5\pi/6$, $3\pi/2$
**71.** 0, $\pi$, $2\pi$ **73.** $0 \leq x \leq \pi/6$ and $5\pi/6 \leq x \leq 2\pi$
**75.** $0 \leq x < \pi/4$, $3\pi/4 < x < 5\pi/4$, $7\pi/4 < x \leq 2\pi$

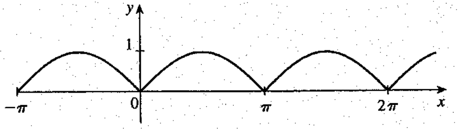
**77.**



**79.**



**81.**



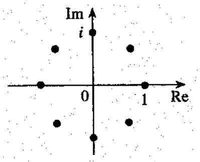
**89.** 14.34457

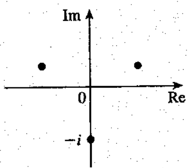
### Exercises G ▪ page A46

**1.** $\frac{1}{3}$ **3.** 0 **5.** (a) $4.82549424 \times 10^{-11}$
**9.** (b) 1, $x^3 + 3x - 4 = 0$

### Exercises H ▪ page A54

**1.** $10 - i$ **3.** $13 - i$ **5.** $12 - 7i$ **7.** $-\frac{1}{2} + \frac{1}{2}i$
**9.** $\frac{1}{2} - \frac{1}{2}i$ **11.** $-i$ **13.** $5i$ **15.** $3 - 4i$, 5 **17.** $4i$, 4
**19.** $\pm\frac{3}{2}i$ **21.** $4 \pm i$ **23.** $-\frac{1}{2} \pm (\sqrt{7}/2)i$
**25.** $3\sqrt{2}[\cos(3\pi/4) + i\sin(3\pi/4)]$
**27.** $5\{\cos[\tan^{-1}\frac{4}{3}] + i\sin[\tan^{-1}\frac{4}{3}]\}$
**29.** $4[\cos(\pi/2) + i\sin(\pi/2)]$, $\cos(-\pi/6) + i\sin(-\pi/6)$, $\frac{1}{2}[\cos(-\pi/6) + i\sin(-\pi/6)]$
**31.** $4\sqrt{2}[\cos(7\pi/12) + i\sin(7\pi/12)]$, $(2\sqrt{2})[\cos(13\pi/12) + i\sin(13\pi/12)]$, $\frac{1}{4}[\cos(\pi/6) + i\sin(\pi/6)]$
**33.** −1024 **35.** $-512\sqrt{3} + 512i$
**37.** ±1, $\pm i$, $(1/\sqrt{2})(\pm 1 \pm i)$ **39.** $\pm(\sqrt{3}/2) + \frac{1}{2}i$, $-i$





**41.** $i$ **43.** $(-1/\sqrt{2}) + (1/\sqrt{2})i$ **45.** $-e^2$
**47.** $\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$, $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$